First-passage-time exponent for higher-order random walks: Using Lévy flights

J. M. Schwarz[1] and Ron Maimon[2]
[1]*Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138*
[2]*Neuman Laboratory, Cornell University, Ithaca, New York 14850*
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We present a heuristic derivation of the first-passage-time exponent for the integral of a random walk [Y. G. Sinai, Theor. Math. Phys. **90**, 219 (1992)]. Building on this derivation, we construct an estimation scheme to understand the first-passage-time exponent for the integral of the integral of a random walk, which is numerically observed to be 0.220 ± 0.001. We discuss the implications of this estimation scheme for the nth integral of a random walk. For completeness, we also address the $n=\infty$ case. Finally, we explore an application of these processes to an extended, elastic object being pulled through a random potential by a uniform applied force. In so doing, we demonstrate a time reparametrization freedom in the Langevin equation that maps nonlinear stochastic processes into linear ones.

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I. INTRODUCTION

We investigate the general random walk $x(t)$ obeying the equation of motion

$$\frac{d^n x(t)}{dt^n}=\eta(t), \tag{1}$$

where η is white noise with zero mean and unit variance and $x(0)=x_0$.

Let us begin with $n=1$. The *first passage time* is the time it takes for the walk to reach zero. When the walk has no bias, as above, there is no definite time to expect such an event and the distribution is a power law. To find the first-passage-time distribution $\rho(t)dt$, start an ensemble of random walkers at $x_0>0$ and at time $t=0$. Whenever a random walk reaches zero, it is removed from the ensemble. Let $P_t(x)$ be the number density of walks at time t and positive x. $P_t(x)$ is a solution of the diffusion equation with absorbing boundary condition, $P_t(0)=0$. More precisely,

$$\begin{aligned} P_t(x)&=\frac{1}{\sqrt{2\pi t}}\{\exp[(x-x_0)^2/2t]-\exp-[(x+x_0)^2/2t]\}\\ &\approx 2x_0\frac{d}{dx}\left(\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}\right) \end{aligned} \tag{2}$$

at long times.

Let $g(t)$ be the integral of this probability distribution over positive values of x. This is the probability that a random walk at time t has not crossed zero. The first-passage-time distribution is then given by

$$\rho(t)dt=-\frac{dg}{dt}dt=\frac{x_0}{\sqrt{2\pi t}}\frac{dt}{t}. \tag{3}$$

Although we will not show it, this result is universal for all symmetric walks. Any random walk that is equally likely to move forward as backward by a given amount has a first-passage-time distribution with the same asymptotic form. This is the Sparre-Anderson theorem [1]. Note that x_0^2 sets the time scale.

The previous discussion is one of the few first-passage-time problems where an exact solution may be easily found by solving a Fokker-Planck equation with absorbing boundaries. Extensions of this method to more complicated random walks, such as the second-order random walk described by

$$\frac{d^2x}{dt^2}=\eta(t), \tag{4}$$

exist [2]. However, we investigate the first-passage-time distribution for this walk, and for any n, in a different way.

At long times, the first-passage-time distribution for these processes is a power law,

$$\rho_n(t)dt\sim\frac{1}{t^{\beta_n}}\frac{dt}{t}. \tag{5}$$

As computed above, $\beta_1=\frac{1}{2}$. β_2 is known to be $\frac{1}{4}$ [3]. The others are most likely not exact fractions and are nontrivial critical exponents of certain statistical models. We first present numerical results for $n=2,3,4,5$. Then, after presenting a heuristic derivation of β_2, we make a quantitative estimate for the shift in first-passage exponent for $n\geqslant3$. Our analytical results will draw heavily from the theory of Levy flights [4]. We also address separately the $n=\infty$ limit [5].

Finally, within zero temperature mean field theory, we demonstrate that the first-passage-time exponent in the $n=2$ case is the avalanche size exponent for a dynamically elastic extended object, like a crack front or interface, being pulled through a random medium by a uniform applied force at a special point in the parameter space [6]. Generically, for these nonequilibrium systems there is a transition from an overall stationary phase to an overall moving phase as the applied force is increased toward a critical value. Increasing the applied force from the static side triggers *local* motion of the interface for some finite amount of time as long as the applied force is held fixed from the time the toppling starts until the time the toppling stops. The distribution of the amount the interface has moved during an ''avalanche'' event, i.e., the avalanche size, gives us information about the continuity (or discontinuity) of the depinning transition and, therefore, has been a focus of study over the years [7].

II. LÉVY FLIGHTS

A *Lévy variable* q_i is a random variable with a power law distribution for large q,

$$P(q)dq \propto \frac{1}{q^{\beta}}\frac{dq}{q}, \tag{6}$$

with $0<\beta<2$. The variance $\langle q^2\rangle$ is infinite for these distributions and β is called the *Lévy exponent* of q.

A *Lévy flight* is a random walk with each step being a Lévy variable. It is the sum of many independently drawn Lévy variables. Because of the infinite variance, a Lévy flight is an irregular walk dominated by the few largest jumps [4].

Let $L_N(Q)$ be the distribution of $Q=\Sigma_{i=1}^{N} q_i$. The Fourier transform $\tilde{L}_N(k)$ is the Nth power of the Fourier transform of $P(q)$. If $\tilde{P}(k)$ were twice differentiable at zero, then $\tilde{L}_N(k) \sim [1-bk^2+O(k^3)]^N \approx e^{-bNk^2}$ for b real and positive. This is the central limit theorem, and it applies when the second moment of a distribution is finite. For a Lévy variable, the second moment of the distribution is infinite, and $\tilde{P}(k)$ has a cusp at zero for $0<\beta<2$. The form of the cusp may be determined as follows: for small k, the Fourier transform is approximately the integral of the distribution over the first wavelength $\lambda=2\pi/k$ or

$$\int_1^{\lambda}\frac{C}{q^{\beta}}\frac{dq}{q}=1-C'\frac{1}{\lambda^{\beta}}=1-C''k^{\beta}. \tag{7}$$

When there is a cusp, the limiting distribution is not a Gaussian, but has the following form [4]

$$\begin{aligned}\tilde{L}_N(k)&=e^{-bN|k|^{\beta}}, \quad k>0\\ \tilde{L}_N(k)&=e^{-b^*N|k|^{\beta}}, \quad k<0.\end{aligned} \tag{8}$$

By rescaling k, b can be made into a pure phase. For the case where $P(q)$ is symmetric about zero, the Fourier transform is real and $b=1$. It is clear that this distribution has the correct form near $k=0$, and increasing N is equivalent to a rescaling of k. The limiting distribution gets wider without changing shape, so it is a fixed point of convolution.

We will need one special nonsymmetric distribution, the distribution of a flight composed of only positive steps. In this case, $L_N(Q)$ is zero for Q negative, therefore $\tilde{L}_N(k)$ must be analytic in the lower half-plane. In other words, $\tilde{L}(k)$ for $k<0$ is the analytic continuation of $\tilde{L}(k)$ for $k>0$. As k is rotated to $-k$ through negative imaginary values, k^{β} acquires a phase $e^{-\pi i\beta}$. We conclude that

$$b^*=be^{-i\pi\beta} \tag{9}$$

$$b=e^{i\pi\beta/2}.$$

For the special case $\beta=\frac{1}{2}$, $b=(1+i)/\sqrt{2}$.

And finally, if we were to compute the first-passage-time distribution for the process

$$\frac{dx}{dt}=\xi(t), \tag{10}$$

where $\xi(t)$ is Lévy noise (a Lévy variable symmetrically distributed about zero), then we would find a first-passage-time exponent of $\frac{1}{2}$. Even though the Lévy flight is irregular, the position does not cross zero any faster than it does in the nearest-neighbor random walk. This is the Sparre-Anderson theorem once more, regarding which we refer the reader to Ref. [1] for the details.

III. NUMERICAL RESULTS

To obtain the first-passage-time exponent β_n numerically is not as easy as it might appear. Direct numerical integration of the equation of motion becomes more cumbersome with increasing n. To efficiently simulate the equation, we have calculated the free-space propagation kernels for $n=2,3,4,5$ (see Appendix). In this context, the propagation kernel is the probability distribution of $x(t_0+t_s)$, given initial values $x(t_0)$, $x'(t_0),\ldots,x^{(n-1)}(t_0)$ and final values $x'(t_0+t_s),\ldots,x^{(n-1)}(t_0+t_s)$. We first generate the highest-order time derivative, $x^{(n-1)}(t_0+t_s)$, and then new values for each lower-order time derivative until $x(t_0+t_s)$ is updated.

The next time step t_s is chosen so that the variance for the next value of x will be smaller. The ratio of the new variance to the old is χ. χ should be small so that the walk does not become negative then positive within one time step. The probability of this occurring is exponentially small in the inverse of χ. With this algorithm, if $x(t_0)$ is large, the time steps are large as well. The number of time steps required for the simulation only grows logarithmically as a function of the first-passage time. Near the end of the simulation, when x is small, the time steps become small as well, and accuracy is not sacrificed.

The first-passage-time distributions for $n=2,3,4,5$ are shown in Fig. 1 on a log-log scale. Each plot contains approximately 10^7 samples in bins of doubling size. Table I contains the linear regression values of the exponents. The results are independent of χ over the range [0.025,0.005], indicating that χ is small enough.

IV. THE FIRST-PASSAGE-TIME EXPONENT FOR $n=2$

For the $n=2$ case, $x(t)$ is the integral of a random walk. In other words, the variable that executes the random walk is $x'(t)$, the velocity. To exploit this fact, we introduce a new time counter i and divide the time axis into intervals Δt_i between the points where the velocity crosses zero. Then the interval sizes Δt_i are first-passage times for an ordinary random walk.

There is one complication. Referring back to Eq. (3), the time scale until a zero-crossing is the initial value squared.

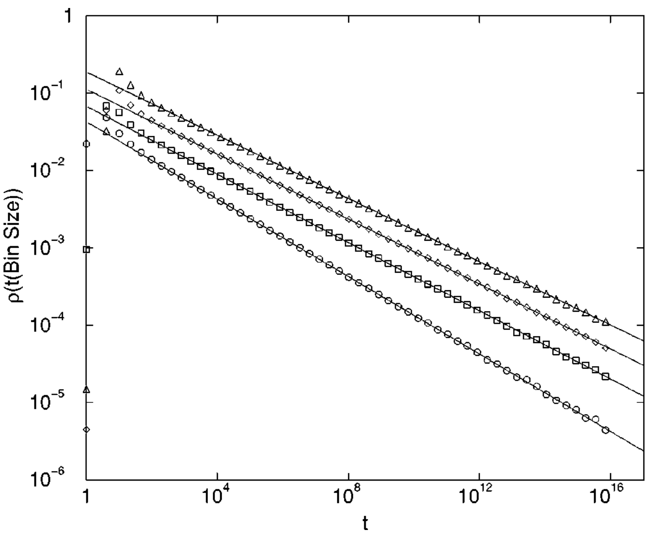

FIG. 1. Log-log plot of the probability of a first-passage time t (in arbitrary units) occurring within the interval $[t/\sqrt{2}, \sqrt{2}t)$, where the probability has been multiplied by a different constant for each curve so that they do not overlap. The open circles denote numerical data for $n=2$; the open squares for $n=3$; the open diamonds for $n=4$; the open triangles for $n=5$. The solid lines represent the results of the linear regression. The size of the symbols is larger than the error bars. As we only used double precision in our simulations, there is an upper cutoff in the first-passage time of approximately 10^{16}.

So right after a zero crossing, the initial value is zero. Since the probability distribution is singular at zero, the random walk reaches zero again instantly, and then again, infinitely many times. This is a well-known property of random walks—they jiggle about every value before moving on. Since this affects only the distribution of the smallest times, we cut off the distribution of times near zero by imagining the system on a lattice. Now we have a finite and discrete set of time intervals Δt_i with each interval distributed with the Levy exponent $\beta_1 = \frac{1}{2}$. These time intervals are independently distributed because the velocity undergoes a simple random walk.

To compute the first-passage time for the position, we must compute the total time $t = \sum_{i=1}^{N} \Delta t_i$ until the position becomes negative. N is the first zero crossing of the velocity that happens at a negative value of the position so we are actually slightly overestimating t by summing N intervals. We can control for this by summing $N-1$ intervals instead, which would be a slight underestimate. This estimate of t, be it over or under, should not affect the first-passage-time exponent since we do not expect the last position step (corresponding to the partial time step) to be arbitrarily large.

TABLE I. The first-passage-time exponents obtained from fits to the data shown in Fig. 1.

n	χ	β_n
2	0.005	0.250 ± 0.001
3	0.005	0.220 ± 0.001
4	0.005	0.210 ± 0.001
5	0.005	0.204 ± 0.001

To continue we need to know two things: (1) the distribution of N, $f(N)$, and (2) given N, the probability distribution of t.

The first problem is actually no problem at all. Since the velocity is just as likely to go up at the start of a time interval as it is to go down, the position is as likely to be positive as negative. Therefore, the position is a symmetric walk on the zero crossings (actually, a symmetric Lévy flight with independent steps distributed with Lévy exponent 1/3 using scaling arguments). By the Sparre-Anderson theorem [1], the distribution of the first-passage N is asymptotically the same as for an unbiased, simple random walk, and

$$f(N) \sim \frac{1}{N^{3/2}} \tag{11}$$

for large N.

And now for the second problem. The distribution of the sum of N-*independent* Δt's, $L_N(t)dt$, is a Lévy distribution. The fact that the steps are distributed with Lévy exponent $\beta_1 = \frac{1}{2}$ and that all of the Δt's are positive fixes the Fourier transform of the Lévy distribution. The inverse Fourier transform can be computed exactly in this case. For large t,

$$L_N(t)dt \sim \frac{N}{t^{1/2}} e^{-N^2/2t} \frac{dt}{t}. \tag{12}$$

This distribution is small for $N^2 > t$, and it may be approximated by $(N/t^{1/2})(dt/t)$ for $N^2 < t$. Using scaling, analogous results may be derived for $\beta \neq \frac{1}{2}$, where a closed form does not exist.

Now, the time intervals we are adding are not really independent. This is because we are restricting attention to Lévy flights that end on the Nth step and this is an atypical sample of all Lévy flights with N steps. However, the properties we actually use from the previous distribution—namely, that it is small for $N^2 > t$ and the appropriate power law for large t—are the scaling laws for the sum of almost any collection of N Lévy variables, where N is large. So, although we do not prove it, we assume that these properties hold for the correct $L_N(t)dt$, or the distribution of the time elapsed for those Lévy flights that end on the Nth step. This is why we refer to our derivation as heuristic.

All the ingredients are now in place for computing β_2. There is a probability $f(N)$ for any value of N, and given N, we know the probability distribution of t. So to find the total distribution of t, we sum up the conditional distributions $L_N(t)$ weighted by the probability of N, or

$$\rho_2(t)dt = \sum_{N=0}^{\infty} f(N) L_N(t) dt. \tag{13}$$

Approximating the sum with an integral and using the approximate scaling form for $L_N(t)$,

$$\rho_2(t)dt \sim \int_0^\infty dN\, f(N) L_N(t) dt \sim \int_0^{t^{1/2}} dN \frac{1}{N^{3/2}} \frac{N}{t^{3/2}} dt. \tag{14}$$

We obtain the following asymptotic form,

$$\rho_2(t)dt = \frac{1}{t^{1/4}} \frac{dt}{t}, \tag{15}$$

so that $\beta_2 = \frac{1}{4}$. This result is unchanged when N is shifted by one unit, so that the last step is of no consequence for the exponent as there are typically many zero crossings, making our derivation self-consistent.

Sinai has rigorously computed β_2 [3]; we have used some similar methods.

V. THE FIRST-PASSAGE-TIME EXPONENT FOR $n \geqslant 3$

The next random walk we consider is the random surge process, governed by

$$\frac{d^3x}{dt^3} = \frac{da}{dt} = \eta(t). \tag{16}$$

The acceleration is now an ordinary random walk. After a time t, the acceleration, velocity and position scale as $a_{\text{typ}} \sim t^{1/2}$, $v_{\text{typ}} \sim t^{3/2}$, and $x_{\text{typ}} \sim t^{5/2}$, respectively. We proceed as we did for the $n=2$ case. Once again, we divide the time axis by the zero crossings of the velocity. The time intervals Δt_i are now distributed with the Lévy exponent $\beta_2 = \frac{1}{4}$. The sum of N-independent Δt's is approximately zero for $t < N^4$, and for larger t, is approximated by

$$L_N(t)dt \sim \frac{N}{t^{1/4}} \frac{dt}{t}. \tag{17}$$

Proceeding casually, we may think that, as before, the distribution $f(N)$ has Lévy exponent of $\frac{1}{2}$ since that result is universal for all symmetric walks. We perform an integral analogous to Eq. (14) and obtain a first-passage-time Lévy exponent $\beta_3 = \frac{1}{8}$. However, our numerical simulations yield a Lévy exponent of 0.220 ± 0.001. The exponent is closer to $\frac{1}{4}$ than $\frac{1}{8}$. Clearly, there is something wrong.

The method fails because there are now *correlations* between the Δx_i steps, where Δx_i is the change in x during a Δt_i step. In the previous $n=2$ case, the velocity undergoes a simple random walk and so its sign is random after a zero crossing. In this $n=3$ case, the velocity is the integral of a random walk. It is a differentiable function and so almost certainly changes sign when it hits zero. Consequently, the Δx_i steps alternate in sign. In addition, as the acceleration drifts about, the Δx_i steps increase in size. The acceleration at the beginning of each next-time interval is larger on average than the previous one, making the next Δt_i and, therefore, the next Δx_i larger than the last.

A large number of alternating, increasing steps will reach zero quickly. From the numerical simulation, with each oscillation there is a definite probability of reaching zero that is almost constant. In other words, the distribution $f(N)$ is not a power law as in the $n=2$ case, but an exponential,

$$f(N) \sim e^{-\lambda_3(2N+1)} \quad \text{with} \quad \lambda_3 = 0.79 \pm 0.02. \tag{18}$$

TABLE II. The first-passage-time exponent estimates for the $n \geqslant 3$, which uses the numerical exponential decay constant λ_n and the numerical results for β_{n-1}. This estimate is to be compared with the last column in Table I.

n	χ	λ_n	β_n^{est}
3	0.005	0.79 ± 0.02	0.199
4	0.005	1.55 ± 0.02	0.210
5	0.005	1.92 ± 0.02	0.205

See Table II and the corresponding Fig. 2. The number of velocity zero crossings must be odd because the position can become negative only after the velocity has become negative. With this exponential distribution, the number of time intervals one must add up is actually quite small. However, adding a finite number of independently drawn Lévy variables distributed with Lévy exponent $\frac{1}{4}$ produces a flight with Lévy exponent $\frac{1}{4}$, so it is quite surprising that adding N of them together, where N has a finite mean, leads to anything other than an exponent of $\frac{1}{4}$.

And yet, adding a small number of *correlated* Lévy variables does shift the exponent. To quantitatively describe the correlation in magnitude, we need to determine how the initial acceleration sets the time scale for each Δt_i interval. We saw from Eq. (3) for the $n=1$ random walk that the time scale is the initial value squared. Therefore, the time scale between two acceleration zero crossings is the initial acceleration squared. Because the time between velocity zero

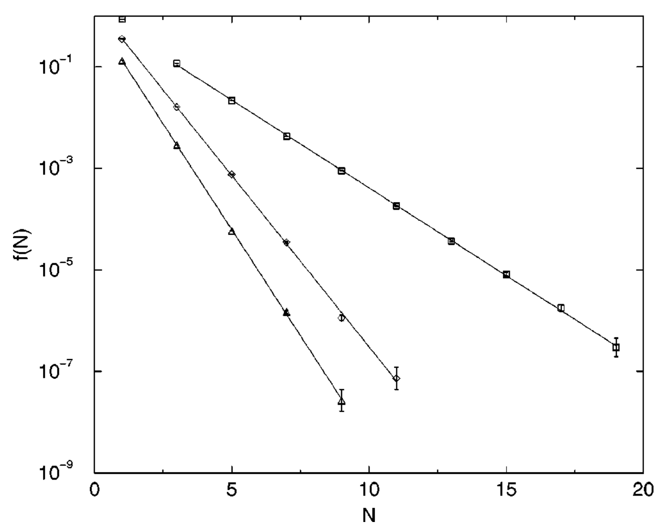

FIG. 2. Log-linear plot of the distribution of number of velocity zero crossings $f(N)$, where $f(N)$ has been multiplied by a different constant for each curve so that they shift and do not overlap. The symbols here are the same as in Fig. 1. For the $n=3$ curve, we ignore the first data point in the linear regression as there is some memory of the initial conditions for this point.

crossings is comprised of many acceleration zero crossings, the time scale between velocity zero crossings is fixed in the same way.

This argument is somewhat general, so it is nice to verify that it is correct. The coefficient in the formula for the first-passage-time distribution was computed in Ref. [2] for the case of the $n=2$ random walk by solving the Fokker-Planck equation with the appropriate boundary conditions.

$$\rho_2(t;v_0,x_0\to 0)dt \sim \frac{\sqrt{v_0}}{t^{1/4}}\frac{dt}{t} = \left(\frac{v_0}{\sqrt{t}}\right)^{1/2}\frac{dt}{t} \tag{19}$$

for large t. Note that $t_{\mathrm{typ}} \sim v_0^2$. Since the acceleration in the $n=3$ case is exactly the same as the velocity in the $n=2$ case, we can translate this result into $n=3$ language by replacing v with a. Therefore in the $n=3$ case, the square of the initial acceleration determines the time scale until the next zero crossing.

So the square of the initial acceleration at each time step determines the scale of the next time step. Since the acceleration is an ordinary random walk, its square has a size proportional to the total elapsed time. Therefore, $\Delta t = Tq$, where q is a unit Lévy variable and T is the total elapsed time. To describe this correlated process, we set the units of time so that Δt_1 is a unit size Lévy variable q_1. The next time step Δt_2 is no longer unit sized, but has a magnitude determined by the square of the acceleration or, equivalently, the total elapsed time. So to find Δt_2 we take Δt_1 and multiply it by separate, independent, unit Lévy variable q_2. To find Δt_3, we take the total elapsed time, $\Delta t_1 + \Delta t_2$, which is the expected square acceleration, and multiply by q_3. In equations,

$$\begin{aligned}
\Delta t_1 &= q_1, \\
\Delta t_2 &= (\Delta t_1) q_2, \\
\Delta t_3 &= (\Delta t_1 + \Delta t_2) q_3, \\
\Delta t_4 &= (\Delta t_1 + \Delta t_2 + \Delta t_3) q_4.
\end{aligned} \tag{20}$$

The total time t that has elapsed after N steps is then

$$t = q_1 \prod_{i=2}^{N} (1+q_i). \tag{21}$$

The asymptotic distribution is extracted when $q_i \gg 1$. This means we need to determine the distribution of products of Lévy variables. Taking the natural logarithm, the product becomes a sum. Define $z_2 = \ln(\Delta t_2)$, $f_1 = \ln(q_1)$, and $f_2 = \ln(q_2)$. We use the following approximate distribution for both Levy variables,

$$\begin{aligned}
P(q)dq &= \frac{\alpha c_0^{\alpha}}{q^{\alpha}}\frac{dq}{q}, \\
P(q)dq &= 0, \quad q < c_0
\end{aligned} \tag{22}$$

with $c_0 = 1$. Keep in mind, in order to compute β_3, the Lévy exponent α of the q_i variables equals $\frac{1}{4}$. In terms of the transformed variables, the distribution of f_1 and f_2 is given by

$$P(f) = \alpha e^{-\alpha f}. \tag{23}$$

If q_1 and q_2 had *different* Lévy exponents, $\alpha_1 > \alpha_2$, the distribution of the sum would be

$$\int_0^{z_2} df_1 P(f_1) P(z_2 - f_1) = \frac{\alpha_1 \alpha_2}{(\alpha_2 - \alpha_1)}(e^{-\alpha_1 z_2} - e^{-\alpha_2 z_2}). \tag{24}$$

For large z_2, $P(z_2) \approx \alpha_2 e^{-\alpha_2 z_2}$, i.e., the smaller exponent dominates the tail. However, when $\alpha_1 = \alpha_2 = \alpha$, we have

$$P(z_2) = \alpha^2 z_2 e^{-\alpha z_2}. \tag{25}$$

In terms of q_1 and q_2, the probability distribution of $q_1 q_2$ is [8]

$$P(\Delta t_2) d(\Delta t_2) = \alpha^2 \frac{\ln(\Delta t_2)}{\Delta t_2^{\alpha}} \frac{d(\Delta t_2)}{\Delta t_2}. \tag{26}$$

In general, the probability distribution for i independent Lévy variables multiplied together is

$$P(z_i) = \frac{\alpha^i}{(i-1)!} z_i^{i-1} e^{-\alpha z_i}. \tag{27}$$

The $(i-1)!$ is required for normalization. Transforming back to the original Δt_i variables,

$$P(\Delta t_i) d(\Delta t_i) = \frac{\alpha^i}{(i-1)!} \frac{(\ln \Delta t_i)^{i-1}}{\Delta t_i^{\alpha}} \frac{d(\Delta t_i)}{\Delta t_i}. \tag{28}$$

We substitute $i = N$ and $\alpha = \frac{1}{4}$ and this distribution becomes $L_N(t)$— the probability distribution of the Nth-passage time of the velocity. Unlike the Nth passage time of the acceleration, $L_N(t)$ is not a Lévy stable distribution because of the correlations. More specifically, $L_N(t)$ is the distribution of a product of N Lévy variables Πq_i. The expression for the Nth passage time we derived earlier is $\Pi(q_i + 1)$, which differs in subleading terms from the previous expression.

Note that $L_N(t)$ coincides with the Nth-passage-time distribution for the position in the $n=2$ case. Both the velocity for $n=3$ and the position for $n=2$ are integrals of an $n=1$ random walk. Observe that the distribution $L_2(t)$ is *not* the same as the distribution of the sum of two variables distributed as $L_1(t)$—there is an additional logarithm. We determined $L_2(t)$ numerically and it agrees with this form (Fig. 3). For higher N, there are $N-1$ factors of $\ln(t)$.

Before estimating β_3, we must make one last argument in the form of an approximation. Up until this point, $L_N(t)dt$ assumes that each step is free to be larger than the previous one. This is because we have not yet taken into account the conditional dependence of the correlation on the position being positive for the walk to persist. In fact, the generic situation when the steps are ever-increasing and alternating in

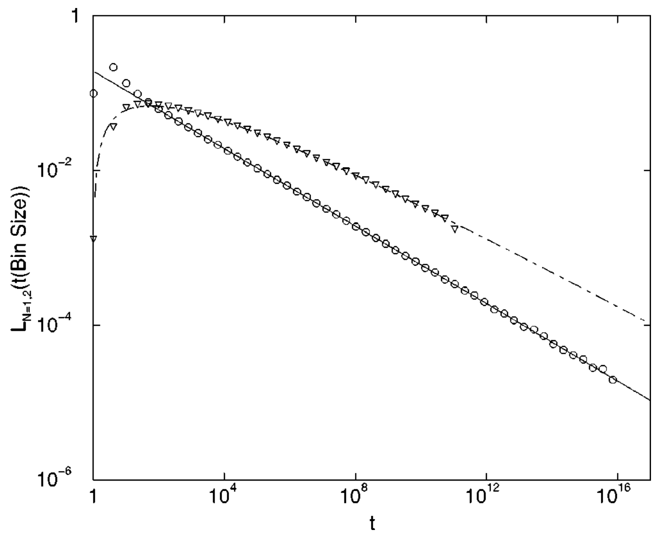

FIG. 3. Log-log plot of $L_1(t)$ and $L_2(t)$ for the $n=2$ case, i.e., the probability of a first (or second) position passage time t (in arbitrary units) occurring within the interval $[t/\sqrt{2},\sqrt{2}t)$. The open circles denote numerical data for $L_1(t)$; the inverted triangles for $L_2(t)$. While the solid line represents the results of the linear regression; the dot-dashed line represents Eq. (28) up to a proportionality constant, with $i=2$ and $\Delta t_i=t$. As we must keep track of even larger absolute values of the position for $N=2$, the double precision constraint cuts off the tail of the distribution even more so for $N=2$ than for $N=1$. Once again, the symbols are larger than the error bars and there are approximately 10^7 samples.

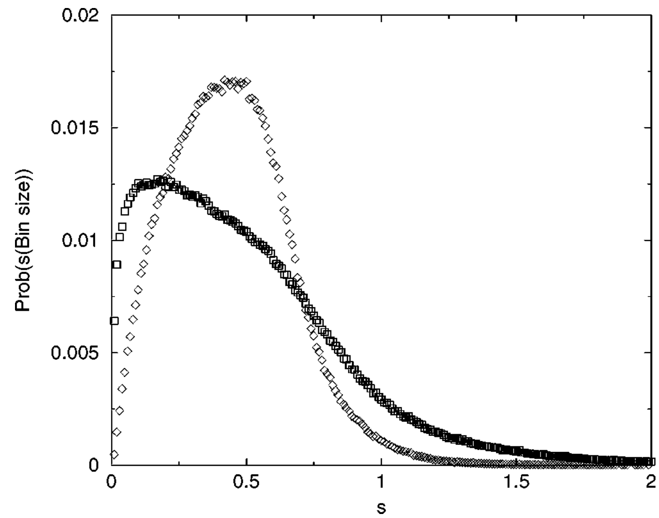

FIG. 4. Plot of the probability of the ratio of the second complete time interval to the first complete time interval, denoted as s, occurring within the interval $[s,s+0.01)$ for the $n=3$ (the open squares) and $n=4$ (the open diamonds) random walk. Both probabilities eventually fall off exponentially as opposed to being distributed as $L_2(\Delta t_2)d(\Delta t_2)$, indicating that the second step is indeed an exceptional pick from the distribution for the first time interval. Each curve contains approximately 10^6 samples.

sign is for there to be only one step. This means that walks that survive more than one step are special. For these walks, the second step is an exceptional pick from the distribution. The *easiest* way for the walk to survive is for the second step to be approximately the same size as the first step. In other words, the size of the first step cuts off the distribution of the second step. The result is that the second step shares the distribution of the first. For the third step, the position is positive, and now its time interval is distributed the same as that of the second step would have been. Generalizing our argument, any two *complete* time intervals can simply be lumped together as one and we conclude that the distribution of the total after N steps is the same as the distribution of the step sizes after $N/2$ steps. In other words, $L_N(t)dt$ is not the correct distribution for N steps, but $L_{N/2}(t)dt$ is. We present evidence for the even-numbered complete steps being typically small in Fig. 4. For $n=3$, the second step is, on average, about five times smaller than the first time interval. For $n=4$, the second time interval is typically twice as small as the first.

So we now have all the ingredients to estimate the Lévy exponent β_3. We arrive at

$$\rho_3(t)dt \sim e^{-\lambda_3}\frac{1}{t^{5/4}}\sum_{N=0}^{\infty}\frac{1}{4^N}\frac{1}{N!}[\ln(t)]^N e^{-2\lambda_3 N}dt \quad (29)$$

$$=e^{-\lambda_3}\frac{1}{t^{5/4}}\{\exp[(e^{-2\lambda_3}/4)]\ln(t)\}dt, \quad (30)$$

which gives $\beta_3^{\rm est}=\frac{1}{4}-(e^{-2\lambda_3}/4)$. Substituting the numerical value $\lambda_3=0.79$, we find the Lévy index 0.199. This result is a shift in the right direction from our original estimate, but it is still too large.

Figure 4 suggests that our estimation scheme may work better for $n>3$ and so we extend the method to $n=4,5,\ldots$. As before, we define the two Lévy flights of the time and position intervals between velocity zero crossings. For $n>3$, the size correlation factor for the time interval Lévy flight turns out to be the same factor of the elapsed time as in the $n=3$ case with each step retaining only the memory of the highest-order derivative. To be more precise, the initial acceleration at the start of each time interval for the $n=3$ case is replaced by the highest-order random variable. We ignore any other memory effects. Then, for the nth-order process, the step-size distribution for the Δt_i intervals is governed by the passage time exponent for the $(n-1)$th-order process. Referring back to Eq. (28), $i=N$ and $\alpha=\beta_{n-1}$. In addition, the correlation in the sign of the Δx_i still persists for the nth-order random walk so the distribution $f(N)$ remains exponential with a decay constant that we determine numerically. Therefore, the nth Lévy exponent will be a small perturbation about the $(n-1)$th first-passage-time exponent with a shift of $-\beta_{n-1}e^{-2\lambda_n}$. For $n=4$ and $n=5$, we observe that our estimation scheme yields more accurate results. See Table II.

As in the $n=2$ case, for $n\geqslant 3$ we consider the last partial time step to be a full one. Given that $L_{N/2}(t)dt$ is the more appropriate distribution, the bulk of the first-passage time for a given N is taken up during the last two steps (up to subleading corrections). So the last partial time steps should be distributed just as the first-passage time. We have numerically verified this to be the case.

Majumdar *et al.* [5] have constructed a completely different approximation scheme to compute the first-passage-time exponent for these higher-order random walks. While we refer the reader to Ref. [5] for the details, their scheme is rather accurate for β_2 and β_3.

VI. THE $n=\infty$ LIMIT

As n increases, we also expect the decay constant λ_n to grow. In fact, in the limit that n becomes large, the first-passage-time exponent reaches a limiting value. We consider the solution to Eq. (1) with initial conditions $x(0)=x'(0)=\cdots=x^n(0)=0$. It is

$$x(t)=\int_0^t dt_n\int_0^{t_n}dt_{n-1}\cdots\int_0^{t_2}dt_1\ \eta(t_1), \tag{31}$$

which is a linear function of η and therefore is the inner product of η with a kernel. In this case,

$$x(t)=\int_0^t dt'\frac{(t-t')^{(n-1)}}{(n-1)!}\eta(t'). \tag{32}$$

It is easy to see that this expression satisfies Eq. (1).

When solving for the time when $x(t)$ first becomes zero, it is permitted to rescale $x(t)$ by any finite number, even if that number is a function of time. This does not change the set of points where $x(t)$ is zero. Rescaling by t^n so that $y(t)=x(t)/t^n$ and dropping constant factors,

$$y(t)=\frac{1}{t}\int_0^t dt'\left(1-\frac{t'}{t}\right)^{n-1}\eta(t'). \tag{33}$$

In the large n limit, $[1-(t'/t)]^n$ is indistinguishable from an exponential in the region where it has the most weight $(t'<nt)$. The result is that

$$y(t)=\frac{1}{t}\int_0^t dt' e^{-nt'/t}\eta(t'). \tag{34}$$

Majumdar *et al.* [5] have noted that the $n=\infty$ random walk has the same first-passage-time exponent as a very different system—the solution of the two-dimensional diffusion equation with a random initial condition,

$$\frac{d\rho}{dt}=\nabla^2\rho,$$
$$\rho(x,y,0)=\eta(x,y). \tag{35}$$

In polar coordinates, the value of $\rho(0,t)$ is given by

$$\rho(0,t)=\frac{1}{2\pi t}\int d^2\vec{r}\, e^{-r^2/2t}\frac{1}{\sqrt{r}}\eta(r,\theta), \tag{36}$$

where we have used $\delta(x)\delta(y)=r^{-1}\delta(r)\delta(\theta)$. The angular integration leaves the integral in terms of a new random variable, $\eta'(r)=(1/\sqrt{2\pi})\int d\theta\,\eta(r,\theta)$. We now have

$$\rho(0,t)=\frac{1}{\sqrt{2\pi t}}\int dr\sqrt{r}e^{-r^2/2t}\eta'(r). \tag{37}$$

Transforming variables to $u=r^2$, we arrive at

$$\frac{1}{2}\frac{1}{t\sqrt{\pi}}\int du\, e^{-u/2t}\eta(u), \tag{38}$$

where we used the fact that $\delta(r)=\delta(u)2\sqrt{u}$ so that $\eta'(r)=\eta(u)\sqrt{2}u^{1/4}$.

Comparing this kernel with the kernel for the $n=\infty$ random walk, we see that they are indeed the same. As a result, the distribution of times for the diffusive field to change sign at one point is the same as the distribution of the first-passage time for the $n=\infty$ random walk for long times, where the initial conditions are irrelevant.

VII. PHYSICAL SYSTEMS

Consider a planar crack front moving through rough, brittle material. In the quasistatic regime, the planar crack front is a line with long-range static elasticity. There is a uniform stress applied to the material, which drives the crack front forward. But there are also local, random variations in the fracture toughness that the crack front needs to overcome to move forward. These randomly varying forces are pinning forces. The competition between the global pulling and local pinning forces as mediated by the elasticity of the crack front determines its dynamics. For small values of the applied force, the pinning forces eventually dominate and the crack front remains stuck. For some finite value of the applied force, the pinning forces are not strong enough to keep the crack front in place and it moves forward. The boundary between these two phases is the depinning transition.

Approaching the depinning transition from the static side, a small increase in the applied force causes a tiny portion of the crack front to move forward. This portion causes a few more regions to move forward as well, only to be eventually stopped by more strongly pinned regions. The only role of the applied force here is to induce the initial motion of the tiny portion. It remains fixed until the local motion of the crack front stops. Near the depinning transition, there exists a sequence of these discrete, localized *avalanche events* with distribution of sizes τ. The avalanche size is defined as the total amount of the crack front that has moved forward during these events. A power law distribution of avalanche sizes as the force approaches the critical force indicates a continuous, second-order-like depinning transition.

Near the depinning transition, the motion of the crack front is jerky with lots of stopping and starting. We therefore model its dynamics in terms of discrete space and time. Within an infinite-range model, where each crack front segment is coupled equally to every other, the spatial degrees of freedom along the crack front are averaged out. We define y_t to be the total number of segments of the crack front that hop forward at discrete time t. Given that we are only concerned with the few segments that are on the verge of hopping forward, y_t is Poissonian distributed. In this infinite-range limit, and in the dissipative regime, the mean of y_t is determined by the number of segments that have hopped at the previous time step only. In addition, the mean of y_t fluctuates statistically until it reaches zero, whereupon the motion stops. In the limit of large avalanches where things are changing slowly with time, the equation of motion for the avalanches at the depinning transition is

$$\frac{dy}{dt}=y^{1/2}\eta(t) \tag{39}$$

in the Ito representation (the time derivatives are forward differences). When $y(t)$ reaches zero, the avalanche is over. We refer the reader to Ref. [6] for a more detailed derivation.

So far, we have neglected any effects like elastic waves that are indeed present along the crack front. If a piece of the crack front moves forward, that motion creates an extra transient force on its neighbors as the elastic wave propagates along the crack front. The extra transient stress is called a stress overshoot. This effect is different than pure inertia, which would be an extra transient stress on the segment itself after it jumps. However, we demonstrate that the two effects are similar in this infinite-range model for a particular value of the stress overshoot [6]. If we were to take into account elastic waves along the crack front, the mean of y_t depends on several previous time steps. In other words, there are higher-order corrections to Eq. (39) that are usually irrelevant in the long time limit. The most significant terms are the first and second time derivatives, and the first time derivative is the only relevant term. If we fine-tune the coefficient of dy/dt to zero, we arrive at a tricritical point. Here, the second-order derivative is the most relevant term and we obtain the equation of motion

$$\frac{d^2y}{dt^2}=y^{1/2}\eta(t). \tag{40}$$

The variable we are interested in is

$$\tau=\int_0^{t_f}dt\, y(t), \tag{41}$$

the avalanche size. We emphasize that t_f is determined by the boundary condition $y(t_f)=0$.

In order to find the distribution of τ, we use the size of the avalanche as a new path-dependent time coordinate. We transform time on each particular path in the path-integral differently, in a way that depends on the history of $y(t)$. This transformation alters the density of $y(t)$'s at any given time, mixing together $y(t)$'s from different times so that they appear simultaneous. Because of this property, it is difficult to perform the transformation in the Fokker-Planck equation; but the transformation is easy in the path integral.

A reparametrization of time, even a path-dependent one, does not affect the answers to questions that do not involve the time explicitly. The probability for a stochastic walker to arrive at point B from point A is independent of the global time, but the probability for a stochastic walker to be at point B at time t is not. Similarly, the avalanche size of a given walk does not involve the time lapse of the walk as the upper cutoff of the integral is determined by the boundary condition on $y(t)$ and not the time explicitly.

Choosing the new time coordinate τ to tick at the new rate,

$$\frac{d\tau}{dt}=y, \tag{42}$$

and using the property of Gaussian noise, $\eta(\tau(t)) =[\eta(t)/\sqrt{(d\tau/dt)}]$, the (Ito) equation pair

$$\frac{dv}{dt}=y^{1/2}\eta(t), \tag{43}$$

$$\frac{dy}{dt}=v$$

become

$$\frac{dv}{d\tau}=\eta(t), \tag{44}$$

$$y\frac{dy}{d\tau}=v.$$

Replacing y with $x=y^2/2$ completes the transformation. The zeros of y are also the zeros of x so the value of τ when x is zero is then the avalanche size. We therefore find the distribution of avalanche sizes to be the distribution of first-passage times τ for the $n=2$ random walk. It has a Lévy exponent $\frac{1}{4}$.

We may perform the same transformation on the Fokker-Planck equation directly, although the motivation then becomes obscure. Consider the following probability-conserving equation,

$$\frac{\partial P}{\partial t}=-y\frac{\partial P}{\partial \tau}+\frac{1}{2}y\frac{\partial^2 P}{\partial v^2}-v\frac{\partial P}{\partial y}, \tag{45}$$

for a new quantity $P_t(x,v,\tau)$. If we integrate $P_t(x,v,\tau)$ over all τ, we recover the original Fokker-Planck equation. On the other hand, if integrate $P_t(x,v,\tau)$ with respect to t from zero to infinity, we obtain

$$\frac{\partial K}{\partial \tau}=\frac{1}{2}\frac{\partial^2 K}{\partial v^2}-\frac{v}{y}\frac{\partial K}{\partial y}, \tag{46}$$

where $K(v,y,\tau)=\int_0^\infty dt\, P_t(v,y,\tau)$. Defining $x=y^2/2$, as before, gives

$$\frac{\partial K}{\partial \tau}=\frac{1}{2}\frac{\partial^2 K}{\partial v^2}-v\frac{\partial K}{\partial x}, \tag{47}$$

which is the Fokker-Planck equation for the $n=2$ random walk. This demonstrates the equivalence of the two problems.

It should now be clear that the avalanche size exponent for Eq. (39) is $\frac{1}{2}$ since the same time reparametrization may be performed in that case also. This relates the avalanche size exponent to the first-passage time of the $n=1$ random walk.

VIII. CONCLUSIONS

We have presented a relation between the shift in the first-passage-time exponent and the decay rate of the probability of N velocity zero crossings for the nth random walk. The method can be exact for $n=2$, but not for $n\geqslant 3$ because of correlations. More work is needed to estimate the decay constant of $f(N)$ since this would determine the convergence rate of the exponents to the $n=\infty$ value. In addition, we do not have any bounds on the error of the first-passage-time exponent as the approximation is uncontrolled. We also do not know which types of correlations for Lévy flights lead to shifts in exponents and which do not. A classification of correlations in terms of these two categories might be useful.

We have opted to slice up time in terms of velocity zero crossings, instead of using the global time in the Fokker-Planck equation. This approach allows us to analyze the higher-order random walks in terms of a one-dimensional Lévy process since the phase variables are the position and the time only. The remaining variables are subsumed in the $(n-1)$th-order random walk that the velocity undergoes between the zero-crossings.

With Eq. (42) we have also reparametrized the global time, but in a different way. We use a path-dependent time transformation to analyze the avalanche statistics of a nonlinear second-order random walk in terms of the first-passage-time exponent of the linear second-order random walk. This allows us to give a novel physical interpretation of the first-passage-time exponent for the $n=2$ case.

Since the avalanche size does not depend on the global time, it has been known to us that one can find the stationary solution of the Fokker-Planck equation corresponding to the nonlinear random walk in Eq. (40) and then arrive at the avalanche size exponent after invoking a simple scaling argument [9]. Since this method gives a new derivation of the avalanche size exponent, it indirectly gives a new derivation of the first-passage-time exponent for the $n=2$ case. This insight may be useful to construct a derivation of β_n for $n\geqslant 3$.

Finally, these higher-order random walks could be relevant for more general forms of dynamic stress transfer along a crack front, or other extended elastic objects. For instance, one can approach a fine-tuned critical point where the coefficients of the first and second time derivative terms are zero. Then the third-order time derivative is the most relevant term, and we arrive at a higher-order critical point. While the time reparametrization given by Eq. (42) does not allow us to equate the avalanche size exponent with the first-passage time-exponent in this case, we remain optimistic that either exponent may be relevant for various physical systems.

Note added in proof. Recently, we were informed of yet another estimation scheme to compute β_n for $n\geqslant 3$ [10].

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APPENDIX: THE KERNEL OF HIGHER-ORDER RANDOM WALKS

We begin with the partition function for all higher-order random walks,

$$Z=\int D[\eta(t)]\exp\left\{-\tfrac{1}{2}\int_0^T[\eta(t)]^2\right\}. \tag{A1}$$

We change variables from $\eta(t)$ to $x(t)$, where $d^nx/dt^n=\eta(t)$, and use forward difference time derivatives. We change variables with the insertion $\int_{x(0)=x_0}^{x(T)=x_f} D[x(t)]\delta([d^nx(t)/dt^n]-\eta(t))$ to arrive at

$$\int_{x(0)=x_0}^{x(T)=x_f} D[x(t)]\exp\left\{-\tfrac{1}{2}\int_0^T[d^nx(t)/dt^n]^2\right\}. \tag{A2}$$

We then minimize the action to find the classical equation of motion with vanishing variations on the boundary for all derivatives up to the $(n-1)$th derivative. In the $n=2$ case we find the equation of motion,

$$\frac{d^4x(t)}{dt^4}=0. \tag{A3}$$

We then impose the following constraints on the $x(t)$ trajectory, $x(0)=x_0$, $x(T)=x_f$, $x'(0)=v_0$, and $x'(T)=v_f$, yielding

$$x(t)=x_0+v_0t+\frac{(3x_f-3x_0-v_fT-2v_0T)}{T^2}t^2+\frac{\left(v_f+v_0-\frac{2}{T}(x_f-x_0)\right)}{T^2}t^3. \tag{A4}$$

The path integral is quadratic, so the partition function is proportional to the exponential of the classical action. Substituting the classical solution into the action and integrating gives the propagation kernel,

$$P_t(v_f,x_f;v_0,x_0)=\frac{\sqrt{3}}{\pi t^2}\exp[-(v_f-v_0)^2/2t]\times\exp\left\{-6\left[(x_f-x_0)-\frac{1}{2}(v_f+v_0)t\right]^2/t^3\right\}. \tag{A5}$$

We normalize the propagation kernel to have integral unity.

Notice that the kernel is the product of a Gaussian in v and a gaussian in x. Given that $x(t)$ is the time integral of $v(t)$ and that the sum of Gaussian variables is Gaussian, if the velocity is Gaussian, so is the position. Not only does the position spread faster than the velocity, but it also drifts with time with a coefficient that depends on the velocity. This method works for higher n giving us the following propagation kernels for $n=3,4,5$:

$$P_t(a_f,v_f,x_f;a_0,v_0,x_0)\sim\frac{1}{t^{9/2}}\exp[-(1/2t)(a_f-a_0)^2]\exp\left\{-(6t^3)\left[(v_f-v_0)-\frac{1}{2}(a_f+a_0)t\right]^2\right\}\times\exp\left\{-(360/t^5)\left[(x_f-x_0)-\frac{1}{2}(v_0+v_f)t+\frac{1}{12}(a_f-a_0)t^2\right]^2\right\} \tag{A6}$$

for $n=3$,

$$P_t(s_f,a_f,v_f,x_f;s_0,a_0,v_0,x_0)\sim\frac{1}{t^8}\exp[-(s_f-s_0)^2/2t]\exp\left\{-(6/t^3)\left[(a_f-a_0)-\frac{1}{2}(s_f+s_0)t\right]^2\right\}\times\exp\left\{-(360/t^5)\left[(v_f-v_0)-\frac{1}{2}(a_f+a_0)t+\frac{1}{12}(s_f-s_0)t^2\right]^2\right\}\times\exp\left\{-(50\,400/t^7)\left((x_f-x_0)-\frac{1}{2}(v_f+v_0)t+\frac{1}{10}(a_f-a_0)t^2-\frac{1}{120}(s_f+s_0)t^3\right)^2\right\} \tag{A7}$$

for $n=4$ with $da(t)/dt=s(t)$, and

$$P_t(u_f,s_f,a_f,v_f,x_f;u_0,s_0,a_0,v_0,x_0)\sim\frac{1}{t^{25/2}}\exp[-(u_f-u_0)^2/2t]\exp\left\{-6/t^3\left[(s_f-s_0)-\frac{1}{2}(u_f+u_0)t\right]^2\right\}\times\exp\left\{-(360/t^5)\left[(a_f-a_0)-\frac{1}{2}(s_f+s_0)t+\frac{1}{12}(u_f-u_0)t^2\right]^2\right\}\times\exp\left\{-(50\,400/t^7)\left[(v_f-v_0)-\frac{1}{2}(a_f+a_0)t+\frac{1}{10}(s_f-s_0)t^2-\frac{1}{120}(u_f+u_0)t^3\right]^2\right\}\times\exp\left\{-(12\,700\,800/t^9)\left[(x_f-x_0)-\frac{1}{2}(v_f+v_0)t+\frac{3}{28}(a_f-a_0)t^2-\frac{1}{84}(s_f+s_0)t^3+\frac{1}{1680}(u_f-u_0)t^4\right]^2\right\} \tag{A8}$$

for $n=5$ with $ds(t)/dt=u(t)$.

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